A talk for the Conference

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On representation numbers by quaternary quadratic forms of level 32 with discriminants 32 and 64

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Plan

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According to a quote attributed to Mr. Eichler, " there are five fundamental operations in arithmetic : addition, subtraction, multiplication, division and modular forms".

Abstract

Abstract

In this paper, we compute explicit the number of integers represented by certain positive-definite, integral, non-diagonal quaternary quadratic forms, namely :

$$x^2 + y^2 + z^2 + 3t^2 + xy + xt,$$
$$x^2 + y^2 + 2z^2 + 2t^2 + xt + yt + 2zt,$$

and

$$x^2 + y^2 + 2z^2 + 3t^2 + xy + xt,$$
$$x^2 + y^2 + z^2 + 8t^2 + xy + xz,$$
$$x^2 + y^2 + 2z^2 + 3t^2 + xt + yt + 2zt.$$

Our approach relies on the theory of modular forms and the use of theta functions of level 32, with discriminants 32 and 64 respectively.

Introduction

For $(x, y, z, t) \in \mathbb{Z}^4$, let $Q := Q(x, y, z, t)$ a positive-definite integral quadratic form and let $n \in \mathbb{N}_0$. We define

$$N(Q(x, y, z, t) = n) := \text{Card}\left\{(x, y, z, t) \in \mathbb{Z}^4 \mid Q(x, y, z, t) = n\right\}$$

The theta function of f is defined by

$$\theta_Q(z) = \sum_{(x,y,z,t)\in\mathbb{Z}^4} q^{Q(x,y,z,t)} = 1 + \sum_{n=1}^{\infty} N\left(Q(x, y, z, t) = n\right) q^n,\ z \in \mathcal{H}.$$

How to determine $N\left(Q(x, y, z, t) = n\right)$?

The solution lies in identifying the coefficients of θ_Q with the elements of the basis of the modular forms space determined by θ_Q.

Some recent works

(2023) Alaca, Ayșe. "Theta functions of nineteen non-diagonal positive-definite quaternary quadratic forms of discriminant 784 with levels 28 or 56." Indian Journal of Pure and Applied Mathematics 54.2, 595-607.

The author explore the theta functions of nineteen positive-definite integral non-diagonal quaternary quadratic forms of discriminant 784 with levels 28 or 56. Also he express these theta functions in terms of Eisenstein series and cusp forms, which we then use to give explicit formulas for the representation number of a positive integer n by their corresponding non-diagonal quaternary quadratic forms.

Some recent works

(2023) Bulent, Kokluce. "REPRESENTATIONS BY QUATERNARY QUADRATIC FORMS WITH COEFFICIENTS 1, 2, 11 AND 22." Bulletin of the Korean Mathematical Society 60.1 : 237-255.

In this article, the author find bases for the spaces of modular forms of level 88 with dirichlet character $\left(\frac{d}{*}\right)$ for $d = 1, 8, 44$ and 88 , also the author derive formulas for the number of representations of a positive integer by the diagonal quaternary quadratic forms with coefficients 1, 2, 11 and 22.

Notations

Let $n \in \mathbb{N}$, we define as :

$$\sigma(n) = \sum_{d|n} d,$$

where d runs through the positive divisors of n. If $n \notin \mathbb{N}$, we set $\sigma(n) = 0$.
Let χ and ψ be a Dirichlet characters of modulus dividing N. For $n \in \mathbb{N}$, the generalized divisor sum $\sigma_{\chi,\psi}(n)$ is defined as :

$$\sigma_{\chi,\psi}(n) := \sum_{1 \leq d|n} \chi(n)\psi\left(\frac{n}{d}\right) d.$$

If $n \notin \mathbb{N}$, we set $\sigma_{\chi,\psi}(n) = 0$. If $\chi = \psi = \chi_1 = 1$ is the trivial dirichlet character, then $\sigma_{\chi,\psi}(n) = \sigma(n)$.
We define the Dirichlet characters by

$$\chi_8(m) = \left(\frac{8}{m}\right), \chi_{-4}(m) = \left(\frac{-4}{m}\right), \chi_{-8}(m) = \left(\frac{-8}{m}\right) \quad (m \in \mathbb{Z})$$

Notations

which are nontrivial Dirichlet characters modulo 8, 4 and 8 respectively. Here, $\left(\frac{d}{m}\right)$ denotes the Legendre-Jacobi-Kronecker symbol.

The theta function of Q is defined by

$$\theta_Q(z) = \sum_{(x,y,z,t)\in\mathbb{Z}^4} q^{Q(x,y,z,t)} = 1 + \sum_{n=1}^{\infty} N\left(Q(x,y,z,t) = n\right) q^n, \ z \in \mathcal{H}.$$

Let $k \in \mathbb{Z}$ and χ be a Dirichlet character of modulus dividing N. Let $M_k(\Gamma_0(N),\chi)$ denote the space of modular forms of weight k with multiplier system χ for $\Gamma_0(N)$, and $E_k(\Gamma_0(N),\chi)$ and $S_k(\Gamma_0(N),\chi)$ denote the subspaces of Eisenstein forms and cusp forms of $M_k(\Gamma_0(N),\chi)$, respectively. It is known that

$$M_k(\Gamma_0(N),\chi) = E_k(\Gamma_0(N),\chi) \oplus S_k(\Gamma_0(N),\chi).$$

The main result

Theorem

Let $n \in \mathbb{N}$. Then

$$N(x^2+y^2+z^2+3t^2+xy+xt=n) = 8\sigma_{(\chi_8,\chi_1)}(n) - 4\sigma_{(\chi_8,\chi_1)}(n/2) + 16\sigma_{(\chi_8,\chi_1)}(n/3) - \sigma_{(\chi_1,\chi_8)}(n) + \sigma_{(\chi_1,\chi_8)}(n/2) - 2\sigma_{(\chi_1,\chi_8)}(n/3) - \sigma_{(\chi_{-4},\chi_{-8})}(n) + 2\sigma_{(\chi_{-8},\chi_{-4})}(n).$$

$$N(x^2+y^2+2z^2+2t^2+xt+yt+2zt=n) = 4\sigma_{(\chi_8,\chi_1)}(n) + 16\sigma_{(\chi_8,\chi_1)}(n/2) - 2\sigma_{(\chi_1,\chi_8)}(n/2).$$

$$N(x^2+y^2+2z^2+3t^2+xy+xt=n) = 1 + 6\sigma(n) - 14\sigma(n/2) + 12\sigma(n/4) - 20\sigma(n/8) + 24\sigma(n/16) - 32\sigma(n/32) - 2\sigma_{(\chi_{-4},\chi_{-4})}(n/2).$$

Theorem

$$N(x^2+y^2+z^2+8t^2+xy+xz=n)=1+9\sigma(n)-29\sigma(n/2)$$
$$+24\sigma(n/4)-44\sigma(n/8)+144\sigma(n/16)-128\sigma(n/32)+3\sigma_{(\chi_{-4},\chi_{-4})}(n)$$
$$+8\sigma_{(\chi_{-4},\chi_{-4})}(n/2)+12a_f(n).$$

$$N(x^2+y^2+2z^2+3t^2+xt+yt+2zt=n)=1+5\sigma(n)-9\sigma(n/2)$$
$$+8\sigma(n/4)-12\sigma(n/8)-16\sigma(n/16)-4a_f(n).$$

where the integers $a_f(n)$ are given by

$$f(z):=\frac{\eta(z)^4\eta(32z)^4}{\eta(2z)^2\eta(16z)^2}=\sum_{n=1}^{\infty}a_f(n)q^n.$$

The key of Proofs

Let χ and ψ be primitive Dirichlet characters with conductors L and R, respectively. The Eisenstein series $E_{k,\chi,\psi}(q)$ is defined as :

$$E_{k,\chi,\psi}(q) = b_0 + \sum_{m\geq 1}\left(\sum_{n|m}\psi(n)\chi\left(\frac{m}{n}\right)n^{k-1}\right)q^m \in \mathbb{Q}(\chi,\psi)[[q]],$$

where

$$b_0 = \begin{cases} 0 & \text{if } L > 1, \\ -\frac{B_{k,\psi}}{2k} & \text{if } L = 1. \end{cases}$$

with $B_{k,\psi}$ denoting the k-th generalized Bernoulli number associated with ψ.

Note that if $\chi = \psi = 1$ and $k \geq 4$, then $E_{k,\chi,\psi} = E_k$, where E_k is the classical Esienstien series.

The key of Proofs

Proposition

Suppose t is a positive integer and χ, ψ be as above and that k is a positive integer such that $\chi(-1)\psi(-1) = (-1)^k$. Except when $k = 2$ and $\chi = \psi = 1$, the power series $E_{k,\chi,\psi}(q^t)$ defines an element of $M_k(RLt, \chi\psi)$. If $\chi = \psi = 1$, $k = 2$, $t > 1$ and $E_2(q) = E_{k,\chi,\psi}(q)$, then $E_2(q) - tE_2(q^t)$ is a modular form in $M_2(\Gamma_0(t))$.

Proposition

The Eisenstein series in $M_k(N, \varepsilon)$ coming from [SW. Theorem 5.8] with RLt divide N and $\chi\psi = \varepsilon$ form a basis for the Eisenstein subspace $E_k(N, \varepsilon)$.

For the space $M_2(\Gamma_0(32), \left(\frac{32}{*}\right))$ the dimension is 8. To compute a basis for this space, it suffices to determine a basis for the Eisenstein subspace $E_2(\Gamma_0(32), \left(\frac{32}{*}\right))$ since we have $\dim(S_2(\Gamma_0(32), \left(\frac{32}{*}\right))) = 0$. Let's calculate a basis for the space $E_2(\Gamma_0(32), \left(\frac{32}{*}\right))$. For the space $E_2(\Gamma_0(32), \left(\frac{32}{*}\right))$ the dimension is 8. Hence,

Proposition

A basis of the space $E_2(\Gamma_0(32), \left(\frac{32}{*}\right))$ is given by :

$$\{E_{\chi_8,\chi_1}(q), E_{\chi_8,\chi_1}(q^2), E_{\chi_8,\chi_1}(q^3), E_{\chi_1,\chi_8}(q), E_{\chi_1,\chi_8}(q^2), E_{\chi_1,\chi_8}(q^3),$$
$$E_{\chi_{-4},\chi_{-8}}(q), E_{\chi_{-8},\chi_{-4}}(q)\}$$

For the space $M_2(\Gamma_0(32))$ the dimension is 8. To compute a basis for this space, we determine a basis for the Eisenstein subspace $E_2(\Gamma_0(32))$. Let us set $L_i = -\frac{1}{24}(E_2(q) - iE_2(q^i))$. Let's calculate a basis for the space $E_2(\Gamma_0(32))$. For the space $E_2(\Gamma_0(32))$ the dimension is 7. Hence,

proposition

A basis of the space $E_2(\Gamma_0(32))$ is given by :

$$\left\{L_2, L_4, L_8, L_{16}, L_{32}, E_{\chi_{-4},\chi_{-4}}(q), E_{\chi_{-4},\chi_{-4}}(q^2)\right\}$$

We use the Sturm bound to determine when two modular forms are equal.

Theorem (Sturm bound)

Let $n \in \mathbb{N}$. If $f(z) = \sum_{n=0}^{\infty} a_f(n)q^n \in M_k(\Gamma_0(N), \chi)$ and $g(z) = \sum_{n=0}^{\infty} a_g(n)q^n \in M_k(\Gamma_0(N), \chi)$ with $a_f(n) = a_g(n)$ for all $n = 0, 1, \ldots, \frac{k}{12}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]$. Then

$$f = g.$$

For $z \in \mathcal{H}$, let $q := e^{2\pi iz}$ so that $|q| < 1$. The Dedekind eta function $\eta(z)$ is defined by

$$\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n).$$

An eta quotient $f(z)$ is a function of the form

$$f(z) := \prod_{\delta \in I} \eta^{r_\delta}(\delta z), \qquad (1)$$

We use the following lemma to determine if certain eta- quotients are modular form.

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et $f(z)$ be an eta quotient given by (1), $k = \frac{1}{2}\sum_{\delta \in I} r_\delta$ and $s = \prod_{\delta \in I} \delta^{r_\delta}$. Suppose that the following conditions are satisfied :

(i) $\sum_{\delta \in I} \delta r_\delta \equiv 0 \pmod{24}$,

(ii) $\sum_{\delta \in I} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}$,

(iii) $\sum_{\delta \in I} \frac{\gcd(d,\delta)^2}{\delta} r_\delta \geq 0$ for each positive integers $d \in I$,

(iv) k is an integer.

Then $f(z) \in M_k(\Gamma_0(N), \chi)$, where the Dirichlet character χ is given by

$$\chi(m) = \left(\frac{(-1)^k s}{m}\right).$$

Theorem

Let $f := f(x_1, \ldots, x_k)$ be a positive-definite integral quadratic form in k variables and $M(f)$ be the matrix of $f(x_1, \ldots, x_k)$. Let N be the level of $f(x_1, \ldots, x_k)$, that is, the least positive integer such that $NM(f)^{-1}$ is an integral matrix with even diagonal entries. Then

$$\theta_f(z) \in M_{\frac{k}{2}}\left(\Gamma_0(N), \chi\right),$$

where the character χ is given by

$$\begin{cases} \left(\frac{2\det(M(f))}{*}\right) & \text{if } k \text{ is odd,} \\ \left(\frac{(-1)^{k/2}\det(M(f))}{*}\right) & \text{if } k \text{ is even.} \end{cases}$$

Base of $S_2(\Gamma_0(32), \binom{64}{*})$

Let's calculate a basis for the space $S_2(\Gamma_0(32))$. Since $\dim S_2(\Gamma_0(42)) = 1$. Thus,

proposition

A basis of the space $S_2(\Gamma_0(32))$ space is given by

$$f(z) := \frac{\eta(z)^4\eta(32z)^4}{\eta(2z)^2\eta(16z)^2} = \sum_{n=1}^{\infty} a_f(n)q^n.$$

Proof of the main theorem

Let

$$Q_1 := Q_1(x, y, z, t) = x^2 + y^2 + z^2 + 3t^2 + xy + xt.$$

The matrix $M(Q_1)$ of Q_1 is

$$\begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 6 \end{bmatrix}$$

so $\det(M(Q_1)) = 32$ and

$$32M(Q_1)^{-1} = \begin{bmatrix} 24 & -12 & 0 & -4 \\ -12 & 22 & 0 & 2 \\ 0 & 0 & 16 & 0 \\ -4 & 2 & 0 & 6 \end{bmatrix}.$$

Proof of the main theorem

Thus, the level of Q_1 is 32 and the character associated with Q_1 is given by

$$\left(\frac{\det(M(Q_1))}{*}\right) = \left(\frac{32}{*}\right)$$

So by Theorem 18 we have

$$\theta_{Q_1}(z) \in M_2\left(\Gamma_0(32), \left(\frac{32}{*}\right)\right).$$

Proof of the main theorem

The Hecke bound of Theorem 16 for $M_2\left(\Gamma_0(32), \left(\frac{32}{*}\right)\right)$ is

$$\left[\frac{2\cdot 32}{12}\prod_{p|32}\left(1+\frac{1}{p}\right)\right] = 8,$$

The first nine terms of θ_{Q_1} are given by

$$\begin{aligned}\theta_{Q_1}(z) =& 1 + 8q + 12q^2 + 12q^3 + 38q^4 + 48q^5 + 24q^6 + 48q^7 + \\ & 78q^8 + O(q^9).\end{aligned} \tag{2}$$

In identifying the coefficients using a SageMath program, agree up to the Sturm bound. Hence, by using Proposition 14 and Theorem 16 with 2, we obtain that

Proof of the theorem

$$\theta_{Q_1}(q) = 8E_{(\chi_8,\chi_1)}(q) - 4E_{(\chi_8,\chi_1)}(q^2) + 16E_{(\chi_8,\chi_1)}(q^3) - E_{(\chi_1,\chi_8)}(q) + E_{(\chi_1,\chi_8)}(q^2) - 2E_{(\chi_1,\chi_8)}(q^3) - E_{(\chi_{-4},\chi_{-8})}(q) + 2E_{(\chi_{-8},\chi_{-4})}(q). \quad (3)$$

By equating coefficients of q^n $(n \in \mathbb{N})$ in 3, we derive

Proof of the main theorem

$$N(x^2+y^2+z^2+3t^2+xy+xt=n) = 8\sigma_{(\chi_8,\chi_1)}(n) - 4\sigma_{(\chi_8,\chi_1)}(n/2)$$
$$+16\sigma_{(\chi_8,\chi_1)}(n/3) - \sigma_{(\chi_1,\chi_8)}(n) + \sigma_{(\chi_1,\chi_8)}(n/2) - 2\sigma_{(\chi_1,\chi_8)}(n/3)$$
$$-\sigma_{(\chi_{-4},\chi_{-8})}(n) + 2\sigma_{(\chi_{-8},\chi_{-4})}(n).$$

Ref 1 A. Alaca and A. Altiary, *Representations by quaternary quadratic forms with coefficients 1, 2, 5 or 10*, Commun. Korean Math. Soc., **34**, 27–41, (2019).
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Thank you !


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